Decoherence induced by squeezing control errors in optical and ion trap holonomic quantum computations

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We study decoherence induced by stochastic squeezing control errors considering the particular implementation of Hadamard gate on optical and ion trap holonomic quantum computers. We find the fidelity for Hadamard gate and compute the purity of the final state when the control noise is modeled by Ornstein-Uhlenbeck stochastic process. We demonstrate that in contradiction to the case of the systematic control errors the stochastic ones lead to decoherence of the final state. In the small errors limit we derive a simple formulae connecting the gate fidelity and the purity of the final state.

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Holonomic quantum computations exploiting non-abelian geometrical phases [1] was primarily proposed in the Ref. [2] and developed further in the Ref. [3]. Many implementations of holonomic quantum computers (HQC) have been proposed. Particularly, the realization of HQC within quantum optics was suggested (optical HQC) [4]. Laser beams in a non-linear Kerr medium were exploited for this purpose. Two different sets of control devices can be used in this case. The first one considered in this Rapid Communication consists of one- and two-mode displacing and squeezing devices. The second one includes SU(2) interferometers. As well trapped ions with the excited state connected to a triple degenerate subspace (four level Λ-system) can be used to implement HQC [5]. Another approach to HQC exploiting squeezing and displacement of the trapped ions vibrational modes was suggested in the Ref. [6]. This implementation of HQC is mathematically similar to the first embodiment of the optical HQC [4] and thus it is also considered in this work. Particularly, expressions for the adiabatic connection and holonomies are the same in these cases. Another proposed implementation of HQC was the HQC with neutral atoms in cavity QED [7]. The coding space was spanned by the dark states of the atom trapped in a cavity. Dynamics of the atom was governed by the generalized Λ-system Hamiltonian. Mathematically similar semiconductor-based implementation of HQC was proposed in the Refs. [8], where one-qubit gates were also realized in the framework of the generalized Λ-system. In distinction from the cavity model of HQC its physical implementation exploits semiconductor excitons driven by sequences of laser pulses [8]. For the two-qubit gate implementation the bi-excitonic shift was used. The generalized Λ-system with the different Rabi frequencies parametrization was exploited recently for HQC implemented by Rf-SQUIDs coupled through a microwave cavity [9]. One more solid state implementa-tion of HQC based on Stark effect was proposed in the Ref. [10].

Let us briefly remind the main results concerning the holonomic quantum computation. In HQC non-abelian geometric phases (holonomies) are exploited to implement unitary transformations over the quantum code. The later is some degenerate subspace C^N spanned on eigenvectors of Hamiltonian H_0, which initiates the parametric isospectral family of Hamiltonians $F = \{H(\lambda) = U(\lambda)H_0 U^\dagger(\lambda)\}_{\lambda \in M}$. Here $U(\lambda)$ is a unitary operator, λ is a vector belonging to the space of the control parameters M and N denotes the dimension of the degenerate computational subspace [2, 3]. Quantum gates are implemented when the control parameters are adiabatically driven along the loops in the control manifold M. The unitary operator mapping the initial state vector belonging to C^N into the final one has the form $e^{i\phi}\Gamma_\gamma(A_\mu)$, where the index μ enumerates control parameters λ_μ constituting vector λ and ϕ is the dynamical phase. Holonomy associated with the loop $\gamma \in M$ is

$$\Gamma_\gamma(A_\mu) = \hat{\mathbf{P}} \exp \left\{ \int_\gamma A_\mu d\lambda_\mu \right\}. \qquad (1)$$

Here $\hat{\mathbf{P}}$ denotes the path ordering operator, A_μ is the matrix valued adiabatic connection given by the expression [1]:

$$(A_\mu)_{mn} = \langle \varphi_m | U^\dagger \frac{\partial}{\partial \lambda_\mu} U | \varphi_n \rangle, \qquad (2)$$

where $|\varphi_k\rangle$ with $k = \overline{1, N}$ are the eigenvectors of the Hamiltonian H_0 forming the basis in C^N. Dynamical phase ϕ will be omitted bellow due to the suitable choice of the zero energy level. We shall consider the single subspace C^N (no energy level crossings are assumed).

It is evident that the quantum gate (holonomy) performed depends on the path passed in the control parameters space. As well it is obvious that in real experiments it is impossible to pass the desired loop in the control manifold without any deviations. Errors in the assign-

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ment of the classical control parameters λ are unavoidable. The question about robustness of holonomic quantum computations with respect to the control errors has attracted a lot of attention recently. Namely, the effect of the errors originated from the imperfect control of classical parameters was studied for $\mathbf{CP}^n$ model of HQC in the Ref. [11] where the control-not and Hadamard gates were particularly considered. Berry phase for the spin $1/2$ particle in a classical fluctuating magnetic field was considered in the Ref. [12]. Approach based on the non-abelian Stokes theorem [13] was proposed in the Ref. [14]. Namely, the general expression for the fidelity valid for arbitrary implementation of HQC in the case of the single systematic control error having arbitrary size and duration was derived. Simple approximate formulae was found in the small error limit. Adiabatic dynamics of quantum system coupled to a noisy classical control field was studied in the Ref. [15]. It was demonstrated that stochastic phase shift arising in the off-diagonal elements of the system's density matrix can cause decoherence. The efficiency of Shor algorithm [16] run on a geometric quantum computer was investigated in the case when the decoherence induced by the stochastic control errors was taken into account. The study of the robustness of the non-abelian holonomic quantum gates with respect to the stochastic fluctuations of the control parameters was presented in the Ref. [17]. Three stability regimes were discriminated in this work for the HQC model with qubits given by polarized excitonic states controlled by laser pulses. Noise cancellation effect for simple quantum systems was considered in the Ref. [18]. Robustness of the parametric family of quantum gates subjected to stochastic fluctuations of the control parameters was studied in the Ref. [19]. Usage of the cyclic states [20] allowed to consider quantum gates which could be continuously changed from dynamic gates to purely geometric ones. It was shown that the maximum of the gate fidelity corresponds to quantum gates with a vanishing dynamical phase. Robust Hadamard gate implementation for optical [4] and ion trap [6] holonomic quantum computers was proposed in the Ref. [21]. The cancellation of the small squeezing control errors up to the fourth order on their magnitude was demonstrated. Hadamard gate is one of the key elements of the main quantum algorithms, for instance see [16, 22]. Thus the search for its robust implementations is of importance.

During the last few years much attention has been payed to the study of both abelian and non-abelian geometric phases in the presence of decoherence which is the most important limiting factor for quantum computations. Let us briefly overview some of these works. The abelian geometric phase of the two-level quantum system interacting with a one and two mode quantum field subjected to the decoherence was considered in the Ref. [23]. It was demonstrated that when the geometric phase is generated by an adiabatic evolution the first correction due to the decoherence of the driving quantized field for the no-jump trajectory has the second or-

der in the decaying rate of the field but it is not the case for the non-adiabatic evolution. Non-abelian holonomies in the presence of decoherence were investigated in the Ref. [24] using the quantum jump approach. The effects of environment on a universal set of holonomic quantum gates were analyzed. Refocusing schemes for holonomic quantum computation in the presence of dissipation were discussed in the Ref. [25]. It has been shown that non-abelian geometric gates realized by means of refocused double-loop scheme possessed a certain resilience against decoherence. Quantum Langevin approach has been used to study the evolution of two-level system with a slowly varying Hamiltonian and interacting with a quantum environment modeled as a bath of harmonic oscillators [26]. It allowed to obtain the dissipation time and the correction to Berry phase in the case of adiabatic cyclic evolution. The realization of universal set of holonomic quantum gates acting on decoherence-free subspaces has been proposed in the Ref. [27]. It has been shown how it can be implemented in the contexts of trapped ions and quantum dots. The performance of holonomic quantum gates in semi-conductor quantum dots under the effect of dissipative environment has been studied in the Ref. [28]. It was demonstrated the influence of the environment modeled by the superhomic thermal bath of harmonic oscillators could be practically suppressed. The study of the non-adiabatic dynamics and effects of quantum noise for the ion trap setup proposed in the Ref. [5] has been also done [29]. The optimal finite operation time was determined. In the references mentioned above the fidelity was used as the main measure of gate resilience.

In this Rapid Communication we consider optical and ion trap implementations of HQC proposed in the Refs. [4] and [6] respectively. Regarding the particular implementation of Hadamard gate we study the decoherence induced by stochastic squeezing control errors. Following the Ref. [12] we model the random fluctuations by Ornstein-Uhlenbeck stochastic process. We analytically obtain the gate fidelity and the final state purity as the measures of the gate robustness with respect to the decoherence induced by stochastic control errors. In the small squeezing control errors limit we derive a simple formulae connecting the gate fidelity and the purity of the final state. As well we demonstrate that systematic control errors do not lead to the decoherence. The systematic error means the error equal for all qubits in the ensemble or for all consecutive gate implementations performed on the given qubit.

One-qubit gates are given as sequence of single mode squeezing and displacing operations [4, 6]:

$$U(\eta, \nu) = D(\eta)S(\nu), \qquad (3)$$

where

$$S(\nu) = \exp\left(\nu a^{\dagger 2} - \bar{\nu} a^2\right),$$
$$D(\eta) = \exp\left(\eta a^{\dagger} - \bar{\eta} a\right) \qquad (4)$$

denote single mode squeezing and displacing operators respectively, $\nu = r_1 e^{i\theta_1}$ and $\eta = x + iy$ are corresponding

complex control parameters, a and $a^\dagger$ are annihilation and creation operators. The line over the parameter denotes complex conjugation. The expressions for the adiabatic connection and the curvature tensor can be found in the Refs. [4, 6]. Following our previous Letter [21] we consider Hadamard gate

$$H_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad (5)$$

implemented when two rectangular loops belonging to the planes $(x, r_1)|_{\theta_1 = 0}$ and $(y, r_1)|_{\theta_1 = 0}$ are passed. Namely,

$$-iH_0 = \Gamma(C_{II})\big|_{\Sigma_{II} = \pi/2}\, \Gamma(C_I)\big|_{\Sigma_I = \pi/4}\,, \qquad (6)$$

where the holonomies are

$$\Gamma(C_I) = \exp\left(-i\sigma_y \Sigma_I\right), \quad \Sigma_I = \int_{S(C_I)} dx\, dr_1\, 2e^{-2r_1},$$

$$\Gamma(C_{II}) = \exp\left(-i\sigma_x \Sigma_{II}\right), \quad \Sigma_{II} = \int_{S(C_{II})} dy\, dr_1\, 2e^{2r_1}, \quad (7)$$

and $S(C_{I,II})$ are the regions in the planes $(x, r_1)|_{\theta_1 = 0}$ and $(y, r_1)|_{\theta_1 = 0}$ enclosed by the rectangular loops C_I and C_{II} respectively. The sides of the rectangles C_I and C_{II} are parallel to the coordinate axes. For the loop C_I these sides are given by the lines $r_1 = 0$, $x = b_x$, $r_1 = d_x$ and $x = a_x$, where the length of the rectangle's sides parallel to the x axis is $l_x = b_x - a_x$. In the Ref. [21] it was shown that

$$d_x = -\frac{1}{2}\ln\left(1 - \frac{\pi}{4l_x}\right), \quad l_x > \frac{\pi}{4}. \qquad (8)$$

In the same way the rectangle C_{II} is composed of the lines $r_1 = 0$, $y = b_y$, $r_1 = d_y$ and $y = a_y$, where [21]:

$$d_y = \frac{1}{2}\ln\left(1 + \frac{\pi}{2l_y}\right), \quad l_y = b_y - a_y. \qquad (9)$$

We restrict ourselves by the consideration of the squeezing control errors only. Moreover, we can neglect the fluctuations of the squeezing control parameter when $r_1 = 0$. Thus to take into account random squeezing control errors we have to replace d_x by $d_x + \delta r_x(x)$ and d_y by $d_y + \delta r_y(y)$, where $\delta r_x(x)$ and $\delta r_y(y)$ are independent Ornstein-Uhlenbeck stochastic processes. Making this substitution into the Eqs. (7) instead of the formulae (6) we obtain the following expression for the perturbed Hadamard gate, see also [21]:

$$-iH = -\frac{1}{\sqrt{2}}\left(\cos\alpha - \sin\alpha\right)\left(\sin\beta + i\sigma_x \cos\beta\right) -$$
$$-\frac{i}{\sqrt{2}}\left(\cos\alpha + \sin\alpha\right)\left(\sigma_z \cos\beta - \sigma_y \sin\beta\right), \quad (10)$$

where

$$\alpha = e^{-2d_x}\int_{a_x}^{b_x} dx \left(1 - e^{-2\delta r_x}\right),$$

$$\beta = e^{2d_y}\int_{a_y}^{b_y} dy \left(e^{2\delta r_y} - 1\right). \qquad (11)$$

Let the qubit initially to be in the pure state $|j\rangle$ with j equal to 0 or 1. Either for the fixed noise realization or in the case of systematic errors the final qubit state will be pure as well. However, it will differ from the desired one. In the real experiment we do not follow the random fluctuations of the control parameters (nevertheless *in principle* we can do it). In this situation quantum mechanics prescribes us to describe the final state of the system by the density matrix and represent the state as a mixture of all possible final states weighted with the probabilities of the corresponding noise realizations. Following this strategy we find the density matrix of the final state for a given noise implementation and than average over the squeezing control parameter fluctuations when the later are modeled by the two independent Ornstein-Uhlenbeck stochastic processes.

Thus for the density operator $\tilde\rho_j \equiv H |j\rangle \langle j| H^\dagger$ we obtain the following matrix elements:

$$\langle j| \tilde\rho_j |j\rangle = \frac{1}{2} + \frac{1}{2}\cos 2\gamma \cos 2\beta,$$

$$\langle nj| \tilde\rho_j |nj\rangle = \frac{1}{2} - \frac{1}{2}\cos 2\gamma \cos 2\beta,$$

$$\langle j| \tilde\rho_j |nj\rangle = \langle nj| \tilde\rho_j |j\rangle^\dagger = \frac{i}{2}\sin 2\beta \cos 2\gamma$$
$$-\frac{1}{2}(-1)^j \sin 2\gamma. \qquad (12)$$

Here $|nj\rangle$ means the state $|not\ j\rangle$, for example, if $j = 0$ than $nj = 1$, the introduced parameter γ is defined as $\gamma = \alpha - \pi/4$. From the Eqs. (12) it immediately follows that $tr\tilde\rho_j = 1$ as it should be.

We assume that the noise δr_x has variance $\tilde\sigma_x$ and a lorentzian spectrum with the bandwidth Γ_x. The fluctuations δr_y have the variance $\tilde\sigma_y$ and bandwidth Γ_y. Using the Eqs. (11)-(12) and properties of Ornstein-Uhlenbeck stochastic process (see Ref. [30]) we average the density matrix $\tilde\rho_j$ over the stochastic fluctuations of the squeezing control parameters δr_x and δr_y. The averaged density matrix $\rho_j = \langle \tilde\rho_j \rangle$ has the following matrix elements:

$$\langle j| \rho_j |j\rangle = \frac{1}{2} + 2e^{-2d_x}l_x \tilde\sigma_x,$$

$$\langle nj| \rho_j |nj\rangle = \frac{1}{2} - 2e^{-2d_x}l_x \tilde\sigma_x,$$

$$\langle j| \rho_j |nj\rangle = \langle nj| \rho_j |j\rangle^\dagger = \frac{(-1)^j}{2}$$
$$-(-1)^j \frac{8\tilde\sigma_x}{\Gamma_x} e^{-4d_x}\left[l_x - \frac{1 - e^{-\Gamma_x l_x}}{\Gamma_x}\right]. \qquad (13)$$

Here we assumed that $\delta r_{x,y} \ll 1$ and restricted ourselves by the first non-vanishing terms depending on δr_x or δr_y. The contribution of the stochastic control errors made in the $(y, r_1)|_{\theta_1=0}$ plane can be neglected compared to the terms appeared due to the errors made in the $(x, r_1)|_{\theta_1=0}$ plane.

Now we find the fidelity of the non-ideal Hadamard gate. In the case when there are no control errors ($\delta r_x = \delta r_y = 0$) the density matrix ρ_{0j} of the final (pure) state has the following matrix elements:

$$\langle j| \rho_{0j} |j\rangle = \langle nj| \rho_{0j} |nj\rangle = \frac{1}{2},$$

$$\langle j| \rho_{0j} |nj\rangle = \langle nj| \rho_{0j} |j\rangle^\dagger = \frac{(-1)^j}{2}. \qquad (14)$$

The non-ideal Hadamard gate fidelity $F \equiv tr(\rho_{0j}\rho_j)$ under the same assumptions as in the Eq. (13) is given by the expression

$$F = 1 - \frac{4\tilde{\sigma}_x}{\Gamma_x l_x}\left(l_x\sqrt{2} - \frac{\pi}{2\sqrt{2}}\right)^2\left[1 - \frac{1 - e^{-\Gamma_x l_x}}{\Gamma_x l_x}\right]. \qquad (15)$$

In our previous work [21] the fidelity was defined as $f = \sqrt{F}$. In the limit $(\Gamma_x l_x)^{-1} \to 0$, when the fluctuations average out, from the Eq. (15) we obtain that $1 - f \sim \tilde{\sigma}_x^2$. It reproduces our previous result [21] concerning the cancellation of the squeezing control errors up to the fourth order on their magnitude (remind that $\tilde{\sigma}_x$ has the order of $(\delta r_x)^2$).

Now we consider decoherence induced by the stochastic squeezing control errors. In order to quantify decoherence strength we exploit the purity of the final state. It is defined as the trace of the squared density matrix. Purity equals to 1 for pure states and less than 1 overwise. From the Eqs. (12) it is easy to obtain that for a fixed noise realization or equivalently in the case of systematic control errors the purity $I_0 = tr\tilde{\rho}_j^2$ equals the unity. Thus the errors equal for all qubits in the ensemble or for all consecutive Hadamard gate implementations performed on a given qubit (systematic control errors) do not lead to the decoherence and the final state remains pure. Nevertheless, fidelity of the gate implementation is less than unity in this case [14, 21].

We use Eqs. (13) to obtain the purity of the final state in the case of the stochastic squeezing control errors. The result can be expressed in a very simple form if we exploit the expression (15) for the gate fidelity:

$$I = tr\rho_j^2 = \frac{1}{2} + \frac{1}{2}\left(1 - 2F\right)^2 \simeq 2F - 1. \qquad (16)$$

The last equality is hold if $(1 - F) \ll 1$. Thus we see that stochastic squeezing control errors induce decoherence and lead the final state to be a mixture of the pure states. Namely, if the fidelity $F < 1$ than the final state purity $I < 1$.

In conclusion, in this Rapid Communication we considered optical and ion trap HQC proposed the Refs. [4] and [6] respectively. Regarding the particular implementation of Hadamard gate we have studied decoherence induced by stochastic squeezing control errors. Ornstein-Uhlenbeck stochastic process was exploited to model random fluctuations of the squeezing control parameter. We have analytically obtained the fidelity of the non-ideal Hadamard gate and found the purity of the qubit's final state. It was shown that the stochastic squeezing control errors reduce the final state into a mixture of pure states and, thus, induce decoherence. In the small errors limit a simple formulae connecting the gate fidelity and the purity of the final state was derived. In contradiction to the case of the stochastic control errors systematic ones do not lead to decoherence and the final state remains pure. Thus systematic control errors lead to wrong output state only whereas stochastic control errors lead both to wrong output and decoherence.

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